# Convergence in inhomogeneous consensus processes with positive diagonals

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#### Abstract

We present a results about convergence of products of row-stochastic matrices which are infinite to the left and all have positive diagonals. This is regarded as in inhomogeneous consensus process where confidence weights may change in every time step but where each agent has a little bit of self confidence. The positive diagonal leads to a fixed zero pattern in certain subproducts of the infinite product.

We discuss the use of the joint spectral radius on the set of the evolving subproducts and conditions on the subproducts to ensure convergence of parts of the infinite product to fixed rank-1-matrices on the diagonal.

If the positive minimum of each matrix is uniformly bounded from below the boundedness of the length of intercommunication intervals is important to ensure convergence. We present a small improvement. A slow increase as quick as  $\log(\log(t))$  in the length of intercommunication intervals is acceptable.

## 1 Introduction

Consider n persons that discuss an issue which can be represented as a real number. Assume further that the persons revise their opinions if they hear the opinions of others. Each person finds his new opinion as a weighted arithmetic mean of the opinions of others. This model of opinion dynamics has been analyzed for the possibilities of consensus by DeGroot [1]. If these weights change over time we have an inhomogeneous consensus process.

While the homogeneous process has strong similarities with a homogeneous Markov chain, things get different when inhomogeneity comes in. While a consensus process relies on row-stochastic matrices multiplied from the left, a Markov process relies on row-stochastic matrices multiplied from the right. And infinity to the right is not the same as infinity to the left.

Consensus processes are only briefly touched in the context of Markov chains [2]. Besides the early approaches of opinion dynamics [1, 3] some results have been made in the context of decentralized computation [4]. Consensus processes fit in the framework of questions about sets of matrices which have the left convergence property 'LCP' [5, 6], which is 'RCP' for transposed matrices.

Recently, there have been independent works that study consensus processes and the underlying matrix-products in the context of opinion dynamics [7, 8, 9], multi agent systems where agents try to coordinate [10, 11] and flocking where birds or robots try to find agreement about their headings [12, 13].

In [12, 14] there have been the first attempts to make the concept of the joint spectral radius work on consensus processes.

In this paper we want to analyze the structure that positive diagonals deliver in inhomogeneous consensus processes and extend the basic idea of [12, 14]. But a result on convergence is only reachable with further assumptions on matrices. In the end we will derive a small improvement on acceptable growth of the length of intercommunication intervals.

## 2 Consensus Processes

For  $n \in \mathbb{N}$  we define  $\underline{n} := \{1, \dots, n\}$ .

Let  $A(0), A(1), \ldots$  be a sequence of square row-stochastic matrices of size  $n \times n$ .

For natural numbers s < t we define a forward accumulation  $A(s,t) := A(s) \dots A(t-1)$  and a backward accumulation  $A(t,s) := A(t-1) \dots A(s)$ . Thus A(s,s+1) = A(s+1,s) = A(s) and A(s,s) is the identity.

Let x(0) be a real column vector of opinions and  $x_i(0)$  stands for the initial opinion of person i. The sequence of vectors x(t) = A(t,0)x(0) is an *inhomogeneous consensus process* and  $a(t)_{ij}$  stands for a confidence weight person i gives to the opinion of agent j at time step t. In this context A(t) is called a *confidence matrix*.

To understand the convergence behavior of inhomogeneous consensus processes the infinite product  $A(\infty, 0)$  is of interest.

In this paper we focus on confidence matrices with positive diagonals. Thus, we regard processes where persons always have a little bit of self-confidence.

A row-stochastic matrix K which has rank 1 and thus equal rows is called a consensus matrix because for a real vector x it holds that Kx is a vector with equal entries and thus represents consensus among persons in a consensus process. Suppose that A(t) := K is a consensus matrix. It is easy to see that for all  $u \ge t$  it holds for the backward accumulation that A(u,0) = K. (For the infinite forward accumulation  $A(0,\infty)$  it only holds that A(0,u) is a consensus matrix but may change with u.) In the following we will point out that there is also a tendency of convergence to consensus matrices.

In the next section we will see that the positive diagonal together with the Gantmacher's canonical form of nonnegative matrices [15] will give us a good overview on the zero and positivity structure of the processes.

In section 5 we go on with a convergence theorem that is built on this structure and conclude in section 6 with a small improvement and discussion on how to fulfill the conditions of the theorem.

## 3 The positive diagonal

We regard two nonnegative matrices A, B to be of the same type  $A \sim B$  if  $a_{ij} > 0 \Leftrightarrow b_{ij} > 0$ . Thus, if their zero-patterns are equal. All matrices of the same type have the same Gantmacher form, which block structure we will outline now.

Let A be a nonnegative matrix with a positive diagonal. For indices  $i, j \in \underline{n}$  we say that there is a  $path \ i \to j$  if there is a sequence of indices  $i = i_1, \ldots, i_k = j$  such that for all  $l \in \underline{k-1}$  it holds  $a_{i_l,i_{l+1}} > 0$ . We say  $i, j \in \underline{n}$  communicate if  $i \to j$  and  $j \to i$ , thus  $i \leftrightarrow j$ . In our case with positive diagonals there is always a path from an index to itself, which we call self-communicating and thus  $i \leftrightarrow j$  is an equivalence relation. An index  $i \in \underline{n}$  is called essential if for every  $j \in \underline{n}$  with  $i \to j$  it holds  $j \to i$ . An index is called inessential if it is not essential.

Obviously,  $\underline{n}$  divides into disjoint self-communicating equivalence classes of indices  $\mathcal{I}_1, \ldots, \mathcal{I}_p$ . Thus, in one class all indices communicate and do not communicate with other indices. The terms essential and inessential thus extend naturally to classes. We define  $n_1 := \#\mathcal{I}_p, \ldots, n_p := \#\mathcal{I}_p$ .

If we renumber indices with first counting the essential classes and second the inessential classes with a class  $\mathcal{I}$  before a class  $\mathcal{J}$  if  $\mathcal{J} \to \mathcal{I}$  then we can bring every row-stochastic matrix A to the *Gantmacher form* [15]

by simultaneous row and column permutations. The diagonal Gantmacher blocks  $A_1, \ldots, A_p$  in (1) are square  $(n_1 \times n_1, \ldots, n_p \times n_p)$  and irreducible. Irreducibility induces primitivity in the case of a positive diagonal. For the non-diagonal Gantmacher blocks  $A_{k,l}$  with  $k=g+1,\ldots,p$  and  $l=1,\ldots,k-1$  it holds that for every  $k \in \{g+1,\ldots,p\}$  at least one block of  $A_{k,1},\ldots,A_{k,k-1}$  contains at least one positive entry.

The spectrum of A is the union of the spectra of all the diagonal Gantmacher blocks.

The following proposition shows that an infinite backward or forward accumulation of nonnegative matrices can be divided after a certain time step into subaccumulations with a common Gantmacher form.

**Proposition 1.** Let  $(A(t))_{t \in \mathbb{N}}$  be a sequence of nonnegative matrices with positive diagonals. Then for the backward accumulation there exists a sequence of natural numbers  $0 < t_0 < t_1 < \dots$  such that for all  $i \in \mathbb{N}$  it holds

$$A(t_{i+1}, t_i) \sim A(t_1, t_0).$$
 (2)

Thus,  $A(t_{i+1}, t_i)$  can be brought to the same Gantmacher form for all  $i \in \mathbb{N}$ . Further on, all Gantmacher diagonal blocks are positive and all nondiagonal Gantmacher-Blocks are either positive or zero.

*Proof.* (In sketch, for more details see [16].)

The proof works with a double monotonic argument on the positivity of entries: While more and more (or exactly the same) positive entries appear in A(t,0) monotonously increasing with rising t, we reach a maximum at  $t_0^*$ . We cut  $A(t_0^*,0)$  of and find  $t_1^*$  when  $A(t,t_0^*)$  reaches maximal positivity again with rising t. We go on like this and get the sequence  $(A(t_{i+1}^*,t_i^*))_{i\in\mathbb{N}}$ . Obviously, less and less (or exactly the same) positive entries appear monotonously decreasing with rising i and we reach a minimum at k. We relabel  $t_j := t_{k+j}^*$  and thus have the desired sequence  $(t_i)_{i\in\mathbb{N}}$  with  $A(t_{i+1},t_i)$  having the same zero-pattern.

Positivity of Gantmacher blocks follows for all blocks  $A(t_{i+1}, t_i)_{[\mathcal{J}, \mathcal{I}]}$  where we have a path  $\mathcal{J} \to \mathcal{I}$ . If we have such a path, then there is a path from each index in  $\mathcal{J}$  to each index in  $\mathcal{I}$  and thus every entry must be positive in a long enough accumulation. Thus, the block has to be positive already, otherwise  $(t_i)_{i \in}$  is chosen wrong.

To prove the result for forward accumulations, we can use the same arguments.  $\hfill\Box$ 

Let us consider now a sequence of row-stochastic matrices  $(A(t))_{t\in\mathbb{N}}$  and their infinite backward products A(t,0) with  $t\to\infty$ . Thus, we face a consensus process where agents may change their confidence weights in every time step.

Form proposition 1 we get the existence of a sequence of time step  $(t_i)_{i\in\mathbb{N}}$  such that all  $A(t_{i+1},t_i)$  have the same Gantmacher form with positive Gantmacher diagonal blocks. So, the Gantmacher structure represents, that agents find a stable confidence structure. There evolve  $g \geq 1$  groups where every agents trust everyone else internally (but maybe indirectly) and no one outside; this repeats for all the time. And there evolve inessential confidence groups in which agents trust each other internal but which also have trust chains to one or more of the g essential groups.

Unfortunately, nothing can be said about the distances  $t_{i+1} - t_i$ .

## 4 The joint spectral radius

We regard a sequence of row-stochastic matrices with positive diagonals  $(A(t))_{t\in\mathbb{N}}$ , take the sequence of time steps of proposition 1 and abbreviate  $A(i) := A(t_{i+1}, t_i)$ . Further on, the  $A_k(i), A_{k,j}(i)$  are the respective Gantmacher blocks of A(i). So,  $\Sigma := \{A(i) | i \in \mathbb{N}\}$  is a set of matrices with the same Gantmacher form, which joint spectral radius can be studied.

The spectral radius of a matrix A is  $\rho(A) := \{ |\lambda| | \lambda \}$  is eigenvalue of  $A \}$  and represents the growth rate of the matrix norm of  $A^i$ . The joint spectral radius [5] of a set of square matrices  $\mathcal{M}$  is

$$\hat{\rho}(\mathcal{M}) := \limsup_{k \to \infty} \sup_{A(i_1), \dots, A(i_k) \in \mathcal{M}} ||A(i_1) \dots A(i_k)||^{\frac{1}{k}}$$

and represents the maximal growth rate of arbitrary products of matrices from  $\mathcal{M}$ .

In our setting for all  $i \in \mathbb{N}$  it holds  $\rho(A(i)) = 1$  and due to the fact that every product of  $\Sigma$  is row-stochastic it holds  $\hat{\rho}(\Sigma) = 1$ , too. But we can do a joint transformation of all matrices in  $\Sigma$  which leads us to a situation where the joint spectral radius is more interesting.

Let us consider the k-th Gantmacher diagonal block  $A_k$  for the essential class  $\mathcal{I}_k$  in an arbitrary accumulation  $A_k(t_{i+1}, t_i) =: A_k(i) \ (k \in \underline{g}, i \in \mathbb{N})$ .  $A(i)_k$  is positive and row-stochastic. Thus, it has the unique maximal eigenvalue 1 for the eigenvector 1. (1 is the vector with only one-entries of the appropriate length given through the context). And there are no other eigenvalues with absolute value one.

According to an idea outlined in [12, 14] we can make a transformation  $P_k A(i)_k P_k^T =: A'(i)$  such that the spectrum stays the same but with eigenvalue 1 removed. For this  $P_k$  is a  $(n_k - 1) \times n$  matrix which rows build an orthogonal basis of the orthogonal complement to span{1}. (This can be normalized vectors with two nonzero entries which have the same absolute value and different signs.)

Thus, A'(i) is  $(n_k - 1) \times (n_k - 1)$ . To see that the spectrum of  $A'(i)_k$  is the spectrum of  $A(i)_k$  without 1 consider an eigenvalue  $\lambda \neq 1$  and one of its eigenvectors x. Then  $y := P_k x$  is not zero and an eigenvector of  $A'(i)_k$  for the eigenvalue  $\lambda$ .  $(A(i)_k x = \lambda x \Rightarrow P_k A(i)_k P_k^T y = \lambda y \Rightarrow A'(i)_k y = \lambda y$ .)

Obviously, all the matrices A(i) have 1 as eigenvalue g times with a g-dimensional eigenspace

$$\operatorname{eig}(A(i), 1) = \operatorname{span}\left\{ \begin{bmatrix} \mathbf{1} \\ 0 \\ \vdots \\ 0 \\ * \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{1} \\ \vdots \\ 0 \\ * \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{1} \\ * \end{bmatrix} \right\}$$
(3)

The \*-parts are not necessary equal for all  $i \in \mathbb{N}$ , but it is clear that 1 is in, thus the \*-parts sum up to multiple of 1.

Nevertheless, we can generalize the transformation idea of [12, 14] to our setting. We define the  $(n-g) \times n$  matrix

$$P := \left[ \begin{array}{ccc} P_1 & & & 0 \\ & \ddots & & \\ & & P_g & \\ 0 & & E \end{array} \right]$$

where E is the unit matrix of size  $n_{g+1} + \cdots + n_p$ . Notice that the blocks are not square and thus not diagonal. Now, it holds  $PA(t_{i+1}, t_i)P^T =$ 

Now we can study the joint spectral radius of  $\Sigma' := PA(i)P^T|i \in \mathbb{N}$ . If we had  $\hat{\rho}(\Sigma') < 1$  this would imply that A(t,0)x(0) would converge in the entries of indices  $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_g$  to a vector in

$$\operatorname{span}\left\{ \begin{bmatrix} \mathbf{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{1} \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{1} \end{bmatrix} \right\}.$$

It holds for the spectral radii that  $\rho(A_1'(i)) < 1, \ldots, \rho(A_g'(i)) < 1$ , due to the the fact that  $A_1(i), \ldots, A_g(i)$  where positive and thus had no other eigenvalues of absolute value one. Further on, the spectral radii of  $A_{g+1}(i), \ldots, A_p(i)$  are less than one because for  $l \in \{g+1, \ldots, p\}$  it holds  $\rho(A_l(i)) \leq ||A_l(i)|| < 1$ . The second inequality holds due to the fact that all row sums in  $A_l(i)$  are less than one.  $(||A|| := \max_i \sum_j |a_{ij}| \text{ in this case.})$ 

Thus, it holds  $\rho(\mathring{A}'(i)) < 1$  for all  $i \in \mathbb{N}$ . But unfortunately this does not imply  $\hat{\rho}(\Sigma') < 1$  [17]. Thus, more assumptions must be made to reach a partial convergence result. This is subject to the next section, where we use concepts of ergodicity.

## 5 Convergence

We define the *coefficient of ergodicity* of a row-stochastic matrix A according to [2] as

$$\tau(A) := 1 - \min_{i,j \in \underline{n}} \sum_{k=1}^{n} \min\{a_{ik}, a_{jk}\}.$$

The coefficient of ergodicity of a row-stochastic matrix can only be zero, if all rows are equal, thus if it is a consensus matrix.

The coefficient of ergodicity is submultiplicative (see [2]) for row-stochastic matrices  $A_0, \ldots, A_i$ 

$$\tau(A_i \cdots A_1 A_0) \le \tau(A_i) \cdots \tau(A_1) \tau(A_0). \tag{4}$$

If  $\lim_{t\to\infty} \tau(A(0,t)) = 0$  we say that A(0,t) is weakly ergodic. Weakly ergodic means that the A(0,t) gets closer and closer to the set of consensus matrices and thus the Markov process gets totally independent of the initial distribution.

For  $M \subset \mathbb{R}_{\geq 0}$  we define  $\min^+ M$  as the smallest positive element of M. For a stochastic matrix A we define  $\min^+ A := \min_{i,j \in \underline{n}}^+ a_{ij}$ . We call  $\min^+$  the positive minimum.

For the positive minimum of a set of row-stochastic matrices  $A_0, \ldots, A_i$  it holds

$$\min^{+}(A_i \cdots A_0) \ge \min^{+}A_i \cdots \min^{+}A_0. \tag{5}$$

**Theorem 2.** Let  $(A(t))_{t\in\mathbb{N}}$  be a sequence of row-stochastic matrices with positive diagonals,  $0 < t_0 < t_1 < \dots$  be the sequence of time steps defined by proposition  $1, \mathcal{I}_1, \dots, \mathcal{I}_g$  be the essential and  $\mathcal{J}$  be the union of all inessential classes of  $A(t_1, t_0)$ .

If for all  $i \in \mathbb{N}$  it holds  $\min^+ A(t_{i+1}, t_i) \geq \delta_i$  and  $\sum_{i=1}^{\infty} \delta_i = \infty$ , then

$$\lim_{t \to \infty} A(t,0) = \begin{bmatrix} K_1 & 0 & 0 \\ & \ddots & \vdots \\ 0 & K_g & 0 \\ \hline & \text{not converging} & 0 \end{bmatrix} A(t_0,0)$$

where  $K_1, \ldots, K_g$  are consensus matrices. (The matrices have to be sorted by simultaneous row and column permutations according to  $\mathcal{I}_1, \ldots, \mathcal{I}_g, \mathcal{J}$ .)

*Proof.* The interesting blocks are the diagonal blocks. It is easy to see due to the lower block triangular Gantmacher form of  $A(t_{i+1}, t_i)$  for all  $i \in \mathbb{N}$ , that all diagonal blocks only interfere with themselves when matrices are multiplied.

Let us regard the essential class  $\mathcal{I}_k$  and abbreviate  $A_i := A(t_{i+1}, t_i)_{|\mathcal{I}_k, \mathcal{I}_k|}$ .

We show that the minimal entry in a column j of a row-stochastic matrix B cannot sink when multiplied from the right with another row-stochastic matrix A,

$$\min_{i \in \underline{n}} (AB)_{ij} = \min_{i \in \underline{n}} \sum_{k=1}^{n} a_{ik} b_{kj} \ge \min_{i \in \underline{n}} b_{ij}.$$

Thus, the minimum of entries in column j of the product  $A_i \cdots A_0$  is monotonously increasing with rising  $i \in \mathbb{N}$ . With similar arguments it follows that the maximum of entries in column j of the product  $A_i \cdots A_0$  is monotonously decreasing with rising  $i \in \mathbb{N}$ .

Further on, it holds due to (4) and the definition of the coefficient of ergodicity that

$$\lim_{i \to \infty} \tau(A_i \dots A_1 A_0) \le \prod_{i=1}^{\infty} \tau(A_i) = \prod_{i=1}^{\infty} (1 - \delta_i) \le \prod_{i=1}^{\infty} e^{-\delta_i} = e^{-\sum_{i=1}^{\infty} \delta_i} = 0.$$

The maximal distance of rows shrinks to zero. Both arguments together imply that  $\lim_{i\to\infty} (A_i \dots A_1 A_0)$  is a consensus matrix which we call  $K_k$ .

Now it remains to show that the  $[\mathcal{J}, \mathcal{J}]$ -diagonal block of the inessential classes converges to zero.

Let us define  $||\cdot||$  as the row-sum-norm for matrices. It holds  $||A_{[\mathcal{J},\mathcal{J}]}(t_{i+1},t_i)|| \le (1-\delta_i)$  and thus like above it holds

$$||A_{[\mathcal{J},\mathcal{J}]}(\infty,t_0)|| \le \prod_{i=1}^{\infty} ||A_{[\mathcal{J},\mathcal{J}]}(t_{i+1},t_i)|| \le \prod_{i=1}^{\infty} (1-\delta_i) \le 0.$$

This proves that  $\lim_{t\to\infty} A_{[\mathcal{J},\mathcal{J}]}(t,0) = 0$ .

An inhomogeneous consensus process A(t,0)x(0) with persons who have some self-confidence stabilizes (under weak conditions) such that we have g consensual subgroups (the essential classes) which have internal consensus, while all other persons (the inessential indices) may hop still around building opinions as convex combinations of the values reached in the consensual groups.

## 6 Discussion on conditions for $\min^+ A(t_{i+1}, t_i) \geq \delta_i$

One thing where theorem 2 stays unspecific is that it demands lower bounds for the positive minimum of the accumulations  $A(t_{i+1}, t_i)$ . But, what properties of the single matrices may ensure the assumption  $\min^+ A(t_{i+1}, t_i) \geq \delta_i$  with  $\sum \delta_i = \infty$ ?

The first idea would be to assume a uniform lower bound for the positive minimum  $\delta < \min^+ A(t)$  for all t. But this is not enough.

Recent independent research [11, 13, 9] has shown that either bounded intercommunication intervals  $(t_{i+1} - t_i < N \text{ for all } i \in \mathbb{N})$  or type-symmetry  $(A \sim A^T)$  of all matrices A(t) can be assumed additional to the uniform lower bound for the positive minimum to ensure the assumptions of theorem 2. But improvements are possible.

Bounded intercommunication intervals Let us regard  $\delta < \min^+ A(t)$  for all  $t \in \mathbb{N}$ . If  $t_{i+1} - t_i \leq N$  it holds by (5) that  $\min^+ A(t_{i+1}, t_i) \geq \delta^N$  and thus  $\sum_{i=0}^{\infty} \delta^N = \infty$  and thus theorem 2 holds. But  $t_{i+1} - t_i$  may slightly rise as the next two propositions show.

**Proposition 3.** Let  $0 < \delta < 1$  and  $a \in \mathbb{R}_{>0}$  then

$$\sum_{n=1}^{\infty} \delta^{a \log(n)} < \infty \iff \delta < e^{-1}. \tag{6}$$

*Proof.* We can use the integral test for the series  $\sum_{n=1}^{\infty} \delta^{a \log(n)}$  because  $f(x) := \delta^{a \log(x)}$  is positive and monotonously decreasing on  $[1, \infty[$ .

With substitution  $y = \log(x)$  (thus  $dx = e^y dy$ ) it holds

$$\begin{split} \int_1^\infty \delta^{a \log(x)} dx &= \int_1^\infty e^{a \log(\delta) \log(x)} dx = \int_1^\infty e^{a \log(\delta)y} e^y dy \\ &= \int_1^\infty e^{ay(\log(\delta)+1)} dy \end{split}$$

The integral is finite if and only if  $\log(\delta) + 1 < 0$  and thus if  $\delta < e^{-1}$ .

**Proposition 4.** Let  $0 < \delta < 1$  and  $a \in \mathbb{R}_{>0}$  then

$$\sum_{n=3}^{\infty} \delta^{a \log(\log(n))} = \infty. \tag{7}$$

*Proof.* We can use the integral test for the series  $\sum_{n=1}^{\infty} \delta^{a \log(\log(n))}$  because  $f(x) := \delta^{a \log(\log(x))}$  is positive and monotonously decreasing on  $[3, \infty[$ . With substitution  $y = \log(\log(x))$  (thus  $dx = e^{(y+e^y)}dy$ ) it holds

$$\begin{split} \int_{3}^{\infty} \delta^{a \log(\log(x))} dx &= \int_{1}^{\infty} e^{a \log(\delta) \log(\log(x))} dx = \int_{1}^{\infty} e^{a \log(\delta) y} e^{y + e^{y}} dy \\ &= \int_{1}^{\infty} e^{a y (\log(\delta) + 1) + e^{y}} dy \end{split}$$

The integral diverges because  $ay(\log(\delta) + 1) + e^y \longrightarrow \infty$  as  $y \to \infty$ .

Thus, assuming  $\min^+ A(t) > \delta > 0$  for all  $t \in \mathbb{N}$  we can allow a slow growing of  $t_{i+1} - t_i$  to fulfill the assumptions of theorem 2. Acceptable is a growing as quick as  $\log(\log(i))$ . If  $t_{i+1} - t_i$  grows as  $\log(i)$  then it must hold  $\delta > e^{-1} > \frac{1}{3}$ . This can only hold if each row of A(t) contains only two positive entries (due to row-stochasticity).

## 7 Conclusion

We pointed out the convergence of the zero patterns of accumulations in inhomogeneous consensus processes with positive diagonals. It leads to a stable Gantmacher form on accumulations. We then extended an idea of [12, 14] to a potential use of the joint spectral radius for the convergence of inhomogeneous consensus processes but saw that further assumptions are necessary to reach a convergence result. For this we switched back to the concept of shrinking coefficients of ergodicity and could reach a small improvement of former results.

Perhaps the combination of both approaches may lead to a full characterization of inhomogeneous consensus processes with respect to convergence and conditions for consensus.

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